Equivariant intersection cohomology of the circle actions*

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To Heisuke Hironaka on the occasion of his 80th birthday.

Abstract

Circle actions on pseudomanifolds have been studied in [10] by using intersection cohomology (see also [4]). In this paper, we continue that study using a more powerful tool, the equivariant intersection cohomology [1, 6].

In this paper, we prove that the orbit space B and the Euler class of the action $\Phi\colon \mathbb{S}^1 \times X \to X$ determine both the equivariant intersection cohomology of the pseudomanifold X and its localization.

We also construct a spectral sequence converging to the equivariant intersection cohomology of X whose third term is described in terms of the intersection cohomology of B.

We consider an action $\Phi\colon \mathbb{S}^1 \times X \to X$ of the circle on a pseudomanifold X whose orbit space B is again a pseudomanifold (cf. (1.1)). We have seen in [10] that the intersection cohomology of X is determined by B and the Euler class $e \in \boldsymbol{H}^2_{\bar{e}}(B)$. In this paper we prove that those two data determine some other structures. The main results of this work are the following:

⇝ The equivariant intersection cohomology[1] $\boldsymbol{H}_{\mathbb{S}^1}(X)$ of X has a Λe-perverse algebra structure[2]. We prove that this structure is determined by B and the Euler class $e \in \boldsymbol{H}^2_{\bar{e}}(B)$ (cf. Proposition 3.2).

⇝ The localization[1] $\boldsymbol{L}_{\mathbb{S}^1}(X)$ of $\boldsymbol{H}_{\mathbb{S}^1}(X)$ has a perverse superalgebra structure. We prove that this structure is determined by B and the Euler class $e \in \boldsymbol{H}^2_{\bar{e}}(B)$ (cf. Proposition 5.2).

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[1] See [1, 6].

[2] Λe $= H^*(\mathbb{CP}^\infty)$.

⇝ For each perversity $\overline{p}$ we construct a spectral sequence converging to $\boldsymbol{H}^{*}_{\overline{p},\mathbb{S}^1}(X)$ whose third term is described in terms of B (cf. Proposition 4.2)[4].

In the last section, we illustrate the results of this work with some particular examples. In the sequel, any manifold will be considered connected, second countable, Haussdorff, without boundary and smooth (of class C^∞).

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1 Modelled actions

We recall in this section some fundamental notions about the objects we deal with. Namely, modelled actions on unfolded pseudomanifolds, the intersection cohomology and the Gysin sequence.

1.1 Unfolded pseudomanifolds ([9], [12])

Let X be a Hausdorff, paracompact and 2nd countable topological space. We say that X is a *stratified space* if it is provided with a stratification $\mathcal{S}$, that is, a finite partition by connected sets called *strata*, which satisfy the following condition: for any two strata $S, S' \in \mathcal{S}$, then $S \cap \overline{S'} \neq \emptyset$ implies $S \subset \overline{S'}$ (in this case, we will write $S \preceq S'$).

Notice that $(\mathcal{S}, \preceq)$ is a partially ordered set, and that a stratum is maximal if and only if it is open. Such a stratum will be said to be *regular*. Every other strata is said to be *singular*. The union Σ of every singular strata is called the *singular part* of the stratified space X. Its complement $X\backslash\Sigma$ is called the *regular part*.

Stratified pseudomanifolds were introduced by Goresky and MacPherson in order to extend the Poincaré duality to stratified spaces. A stratified space X is a *stratified pseudomanifold* if for every stratum S of X there exists a family of *charts*, i.e., embeddings $\alpha\colon U_\alpha \times c(L_S) \to X$ such that:

- $c(L_S)$ is the cone of a compact stratified space L_S called the *link* of S;
- $\{U_\alpha\}_\alpha$ is an open cover of S;
- $\alpha(u, *) = u$ for every $u \in U_\alpha$, being $*$ the apex of the cone $c(L_S)$.

Now, we recall the notion of an unfolding, which we use in order to define the intersection cohomology of a stratified pseudomanifold by means of differential forms. We will say that a stratified pseudomanifold X is an *unfolded pseudomanifold* if it admits an *unfolding*, which consists of a manifold $\widetilde{X}$, a surjective, proper continuous function $\mathscr{L}\colon \widetilde{X} \to X$ and a family of unfoldings $\mathscr{L}_L\colon \widetilde{L} \to L$ of the links of the strata of X satisfying:

1. the restriction $\mathscr{L}\colon \mathscr{L}^{-1}(X\backslash\Sigma) \longrightarrow X\backslash\Sigma$ is a smooth trivial finite covermap;
2. $\mathscr{L}^{-1}(\Sigma)$ is covered by unfoldable charts α, i.e., we have a commutative diagram

$$\begin{tikzcd} U \times \widetilde{L} \times \mathbb{R} \arrow[r, "\widetilde{\alpha}"] \arrow[d, "c"] & \widetilde{X} \arrow[d, "\mathscr{L}"] \\ U \times c(L) \arrow[r, "\alpha"] & X \end{tikzcd}$$

where $\widetilde{\alpha}$ is a diffeomorphism onto $\mathscr{L}^{-1}(\mathrm{Im}(\alpha))$ and the left vertical map is given by $c(u,x,t) = (u, [\mathscr{L}_L(x), |t|])$.

[4] As C. Allday pointed out to us, this spectral sequence degenerates into the Skjelbred exact sequence of [13] when $\overline{p} = \overline{0}$ (cf. Proposition 4.5).

1.2 Perversities and intersection cohomology ([12, sec. 3])

Recall that if $\pi\colon M \to B$ is a surjective submersion, the *perverse degree* $||\omega||_B$ of a differential form $\omega \in \Omega^*(M)$ is the smallest integer m such that $i_{\xi_0} \dots i_{\xi_m}\omega = 0$ for every collection of vector fields $\xi_0, \dots, \xi_m$ tangent to the fibers of π. By convention, $||0||_B = -\infty$.

Denote by $\mathcal{S}^{\rm sing}$ the set of singular strata of the stratified pseudomanifold X. Recall that a *perversity* $\overline{p}$ in X is a map $\overline{p}\colon \mathcal{S}^{\rm sing} \to \mathbb{Z}$. We denote by $\mathcal{P}_X$ the set of all perversities of X. Fix an unfolding $\mathscr{L}\colon \widetilde{X} \to X$. We say that a form $\omega \in \Omega^*(X\backslash\Sigma)$ is *liftable* if there exists a form $\widetilde{\omega} \in \Omega^*(\widetilde{X})$ such that $\mathscr{L}^*(\omega) = \widetilde{\omega}$. The algebra of liftable forms of X is denoted by $\Pi^*(X)$. Given a perversity $\overline{p}$ in X, recall that the cohomology of the complex

$$\Omega^*_{\overline{p}}(X) = \{\omega \in \Pi^*(X) : \ ||\omega||_S \leq \overline{p}(S) \text{ and } ||d\omega||_S \leq \overline{p}(S), \ \forall S \in \mathcal{S}^{\rm sing}\}$$

is the $\overline{p}$-*intersection cohomology* of X, and is denoted by $\boldsymbol{H}^*_{\overline{p}}(X)$.

1.3 Modelled action ([9, sec. 4], [10, sec. 1.1])

Under some assumptions, the orbit space of an action of the circle on a stratified pseudomanifold is also a stratified pseudomanifold, which is called $\mathbb{S}^1$-pseudomanifold in [11, sec. 4]. In this work we shall use a variant of this concept: modelled actions of $\mathbb{S}^1$ on unfolded pseudomanifolds. We list below the main properties of a modelled action $\Phi\colon \mathbb{S}^1 \times X \longrightarrow X$ of the circle $\mathbb{S}^1$ on an unfolded pseudomanifold X. We denote by $\pi\colon X \to B$ the canonical projection onto the orbit space $B = X/\mathbb{S}^1$.

(MA.i) *The isotropy subgroup $\mathbb{S}^1_x$ is the same for each $x \in S$. It will be denoted by $\mathbb{S}^1_S$.*

(MA.ii) *For each regular stratum R we have $\mathbb{S}^1_R = \{1\}$.*

(MA.iii) *For each singular stratum S with $\mathbb{S}^1_S = \mathbb{S}^1$, the action Φ induces a modelled action $\Phi_{L_S}\colon \mathbb{S}^1 \times L_S \to L_S$, where L_S is the link of S.*

(MA.iv) *The orbit space B is an unfolded pseudomanifold, relatively to the stratification $\mathcal{S}_B = \{\pi(S) \ / \ S \in \mathcal{S}_X\}$, and the projection $\pi\colon X \to B$ is an unfolded morphism.*

(MA.v) *The assignment $S \mapsto \pi(S)$ induces the bijection $\pi_S\colon \mathcal{S}_X \to \mathcal{S}_B$.*

The action Φ may induce two kind of strata of X:

- a stratum S is *mobile* when $\mathbb{S}^1_S$, the isotropy subgroup of any point of S, is finite and
- a stratum S is is *fixed* when $\mathbb{S}^1_S$, the isotropy subgroup of any point of S, is $\mathbb{S}^1$.

Recall that the regular stratum R is mobile with $\mathbb{S}^1_R = \{1\}$. In this work, we need the refinement of fixed strata introduced in [9, sec. 5.6]

- a fixed stratum S is *perverse*[5] when the Euler class of the action $\Phi_{L_S}\colon \mathbb{S}^1 \times L_S \to L_S$ does not vanish, where L_S is the link of S.

1.4 Gysin sequence ([9, sec. 6], [10, sec. 1.3])[6]

[5] See [10, sec. 1.2] for some examples. Notice that, as G. Friedman pointed out in [3] , there is a misprint in [10, sec. 1.1] in the definition of perverse stratum: it should be $H^*(L_S\backslash\Sigma_{L_S}) \neq H^*\left((L_S\backslash\Sigma_{L_S})/\mathbb{S}^1\right) \otimes H^*\left(\mathbb{S}^1\right)$, where Σ_{L_S} is the singular part of the link L_S. That definition is equivalent to the one we give above.

[6] The Gysin sequence for intersection cohomology has been constructed in [9, sec. 6]. In this article we use the notations of [10, sec. 1.3]. Notice that, as G. Friedman pointed out in [3], in the definition of $\mathcal{G}^*_{\overline{p}}(B)$ given in [10, sec. 1.3], the degree should be shifted by 1, so that $\mathcal{G}^*_{\overline{p}}(B) \subset \Omega^*_{\overline{p}-\overline{\chi}}(B)$.

Fix a modelled action of $\mathbb{S}^1$ on X. Recall that the complex of *invariant $\overline{p}$-forms* computes the $\overline{p}$-intersection cohomology of X. This complex can be described in terms of basic data as follows: consider the graded complex

$$\text{(1)} \qquad \underline{\Omega}^*_{\overline{p}}(X) = \left\{ (\alpha,\beta) \in \Pi^*(B) \oplus \Omega^{*-1}_{\overline{p}-\overline{x}}(B) \middle/ \begin{bmatrix} \|\alpha\|_{\pi(S)} \leq \overline{p}(S) \\ \|d\alpha + (-1)^{|\beta|}\beta \wedge \epsilon\|_{\pi(S)} \leq \overline{p}(S) \end{bmatrix} \text{ if } S \in \mathcal{S}^{sing}_X \right\}$$

endowed with the differential $D(\alpha,\beta) = (d\alpha + (-1)^{|\beta|}\beta \wedge \epsilon, d\beta)$. Here $|-|$ stands for the degree of the form, $\epsilon \in \Pi^2(B)$ is an Euler form (i.e., $\epsilon = d\chi$ for a characteristic form χ of the action) and $\overline{x}$ is the characteristic perversity defined by $\overline{x}(\pi(S)) = \begin{cases} 1 & \text{if } S \text{ is a fixed stratum} \\ 0 & \text{if } S \text{ is a mobile stratum.} \end{cases}$

The assignment $(\alpha,\beta) \mapsto \pi^*\alpha + \pi^*\beta \wedge \chi$ establishes a differential graded isomorphism between $\underline{\Omega}^*_{\overline{p}}(X)$ and $\left(\Omega^*_{\overline{p}}(X)\right)^{\mathbb{S}^1}$. This gives rise to the long exact *Gysin sequence*:

$$\cdots \longrightarrow \boldsymbol{H}^{i+1}_{\overline{p}}(X) \xrightarrow{\oint_{\overline{p}}} H^i\left(\mathcal{G}^*_{\overline{p}}(B)\right) \xrightarrow{\mathbf{e}_{\overline{p}}} \boldsymbol{H}^{i+2}_{\overline{p}}(B) \xrightarrow{\pi_{\overline{p}}} \boldsymbol{H}^{i+2}_{\overline{p}}(X) \longrightarrow \ldots,$$

where the *Gysin term* $\mathcal{G}^*_{\overline{p}}(B)$ is the differential complex

$$\left\{ \beta \in \Omega^{*-1}_{\overline{p}-\overline{x}}(B) \middle/ \exists \alpha \in \Pi^*(B) \text{ with } \begin{bmatrix} \|\alpha\|_{\pi(S)} \leq \overline{p}(S) \text{ and} \\ \|d\alpha + (-1)^{|\beta|}\beta \wedge \epsilon\|_{\pi(S)} \leq \overline{p}(S) \end{bmatrix} \text{ if } S \in \mathcal{S}^{sing}_X \right\}.$$

Recall that the Euler perversity $\overline{e}$ is defined by $\overline{e}(S) = \begin{cases} 0 & \text{when } S \text{ mobile stratum,} \\ 1 & \text{when } S \text{ not perverse fixed stratum} \\ 2 & \text{when } S \text{ perverse stratum.} \end{cases}$

So, the Euler class $e = [\epsilon]$ belongs to $\boldsymbol{H}^2_{\overline{e}}(B)$.

1.5 Perverse algebras ([10, subsection 2.2])

A *perverse set* is a triple $(\mathcal{P},+,\leq)$ where $(\mathcal{P},+)$ is an abelian semi-group with unit element $\overline{0}$ and $(\mathcal{P},\leq)$ is a partially ordered set such that $\leq$ and $+$ are compatible. Notice that the set of all perversities of an unfolded pseudomanifold $\mathcal{P}_X$ is a perverse set.

Recall that a *differential graded commutative (dgc, for short) perverse algebra* (or simply a perverse algebra) is a quadruple $\boldsymbol{E} = (E,\iota,\wedge,d)$ where

- $E = \bigoplus_{\overline{p}\in\mathcal{P}} E_{\overline{p}}$ where each $E_{\overline{p}}$ is a graded (over $\mathbb{Z}$) vector space,
- $\iota = \left\{\iota_{\overline{p},\overline{q}} \colon E_{\overline{p}} \to E_{\overline{q}} \;/\; \overline{p} \leq \overline{q}\right\}$ is a family of graded linear morphisms, and
- $(E,d,\wedge)$ is a dgc algebra,

verifying

i) $\iota_{\overline{p},\overline{p}}$ = Identity	*ii)* $\iota_{\overline{q},\overline{r}} \circ \iota_{\overline{p},\overline{q}} = \iota_{\overline{p},\overline{r}}$	*iii)* $\wedge\left(E_{\overline{p}} \times E_{\overline{p}'}\right) \subset E_{\overline{p}+\overline{p}'}$
iv) $d\left(E_{\overline{p}}\right) \subset E_{\overline{p}}$	*v)* $\iota_{\overline{p}+\overline{p}',\overline{q}+\overline{q}'}(a \wedge a') = \iota_{\overline{p},\overline{q}}(a) \wedge \iota_{\overline{p}',\overline{q}'}(a')$	*vi)* $d \circ \iota_{\overline{p},\overline{q}} = \iota_{\overline{p},\overline{q}} \circ d$

Here, $\overline{p} \leq \overline{q} \leq \overline{r}$, $\overline{p}' \leq \overline{q}'$, $a \in E_{\overline{p}}$ and $a' \in E_{\overline{p}'}$.

For example, associated to a modelled action, the following dgc algebras have the structure of perverse algebras: $\boldsymbol{\Omega}(X) = \bigoplus_{\overline{p}\in\mathcal{P}_X} \Omega_{\overline{p}}(X)$ and $\boldsymbol{H}(X) = \bigoplus_{\overline{p}\in\mathcal{P}_X} \boldsymbol{H}_{\overline{p}}(X)$.

Remark 1.6. Notice that the intersection cohomology relatively to a fixed perversity $\overline{p}$ is not an algebra, due to property *iii*), that is, $\bigwedge\colon \boldsymbol{H}^i_{\overline{p}}(B) \times \boldsymbol{H}^j_{\overline{q}}(B) \longrightarrow \boldsymbol{H}^{i+j}_{\overline{p}+\overline{q}}(B)$. With perverse algebras, we recover the algebra structure, as we consider all perversities together. The category of these objects has been studied recently in [5]. The perverse algebra structure has been used, too, in [2] to extend the theory of minimal models to the context of intersection cohomology. We expect to obtain analogous results in the context of the present article.

Remark 1.7. For some specific contexts such as pseudomanifolds arising from complex algebraic varieties, it would be more natural to work just with the middle perversity $\overline{m}$. Nevertheless, as follows from the previous remark, multiplication by the Euler class $e \in \boldsymbol{H}^2_{\overline{e}}(B)$ does not define an endomorphism of $\boldsymbol{H}^*_{\overline{m}}(B)$ unless all strata are mobile, but a homomorphism $\boldsymbol{H}^*_{\overline{m}}(B) \to \boldsymbol{H}^{*+2}_{\overline{m}+\overline{e}}(B)$ instead. So, to work with this homomorphism in a suitable category, we need to work with all possible perversities together, which leads us to perverse algebras.

2 Equivariant intersection cohomology

We introduce in this section the equivariant intersection cohomology [1, 6] of a modelled action [9]. For the rest of this work, we fix a modelled action $\Phi\colon \mathbb{S}^1 \times X \to X$. We denote by B the orbit space $X/\mathbb{S}^1$.

2.1 Equivariant intersection cohomology. We fix $\overline{p}$ a perversity of X. As $\mathbb{S}^1$ is connected and compact, the cohomology of the subcomplex of $\mathbb{S}^1$-invariant forms $\underline{\Omega}^*_{\overline{p}}(X)$ is $\boldsymbol{H}^*_{\overline{p}}(X)$.

Recall that the classifying space of $\mathbb{S}^1$ is just $\mathbb{CP}^\infty$ whose cohomology is the free dgc algebra $\Lambda\mathsf{e}$ where $|\mathsf{e}| = 2$ and $d\mathsf{e} = 0$. The *equivariant intersection cohomology* $\boldsymbol{H}^*_{\overline{p},\mathbb{S}^1}(X)$, relatively to the perversity $\overline{p}$, is the cohomology of the complex $\left(\underline{\Omega}^*_{\overline{p}}(X) \otimes \Lambda\mathsf{e}, \nabla\right)$, where ∇ is defined linearly from

$$\nabla((\alpha,\beta)\otimes \mathsf{e}^n) = D(\alpha,\beta)\otimes \mathsf{e}^n + (-1)^{|\beta|}(\beta,0)\otimes \mathsf{e}^{n+1}.$$

The equivariant intersection cohomology generalizes the usual equivariant cohomology since $\boldsymbol{H}^*_{\overline{0},\mathbb{S}^1}(X) = H^*_{\mathbb{S}^1}(X)$ when X is normal.

2.2 $\Lambda\mathsf{e}$-perverse algebras. We have introduced in [10, sec. 2] the notion of perverse algebra, perverse morphism and perverse isomorphism. The coefficient ring of these objects is $\mathbb{R}$. When we replace this ring by $\Lambda\mathsf{e}$, we get the notions of $\Lambda\mathsf{e}$-*perverse algebra*, $\Lambda\mathsf{e}$-*perverse morphism* and $\Lambda\mathsf{e}$-*perverse isomorphism*. In this work, we deal with the following examples:

+ The quadruple $\boldsymbol{\Omega}_{\mathbb{S}^1}(X) = \left(\bigoplus_{\overline{p}\in\mathcal{P}_X} \underline{\Omega}_{\overline{p}}(X)\otimes\Lambda\mathsf{e}, \iota, \wedge, \nabla\right)$ is a $\Lambda\mathsf{e}$-perverse algebra. Here, the $\Lambda\mathsf{e}$-structure is given by: $\mathsf{e}\cdot((\alpha,\beta)\otimes\mathsf{e}^n) = (\alpha,\beta)\otimes\mathsf{e}^{n+1}$.
+ Its cohomology $\boldsymbol{H}_{\mathbb{S}^1}(X) = \left(\bigoplus_{\overline{p}\in\mathcal{P}_X} \boldsymbol{H}_{\overline{p},\mathbb{S}^1}(X), \iota, \wedge, 0\right)$ is the the *equivariant intersection cohomology algebra* which is a $\Lambda\mathsf{e}$-perverse algebra.
+ The operator $\boldsymbol{\pi}'\colon \boldsymbol{H}(B)\otimes\Lambda\mathsf{e} \to \boldsymbol{H}_{\mathbb{S}^1}(X)$, defined by $\pi'_{\overline{p}}([\alpha]\otimes\mathsf{e}^n) = [(\alpha,0)\otimes\mathsf{e}^n]$, is a $\Lambda\mathsf{e}$-perverse morphism.

Remark 2.3. The Λe-perverse morphism π' suggests that the equivariant perverse minimal model of X (see Remark 1.6) may be computed by the mimimal model of B, as it happens in the case without perversities.

2.4 Equivariant Gysin sequence. The main tool we use for the classification of modelled actions is the Gysin sequence we construct now. Fix $\overline{p}$ a perversity of X. Consider the short exact sequence

$$0 \to \left(\Omega^*_{\overline{p}}(B) \otimes \Lambda \mathrm{e}, d \otimes 1\right) \xrightarrow{\pi'_{\overline{p}}} \left(\underline{\Omega}^*_{\overline{p}}(X) \otimes \Lambda \mathrm{e}, \nabla\right) \xrightarrow{\oint'_{\overline{p}}} \left(\mathcal{G}^{*-1}_{\overline{p}}(B) \otimes \Lambda \mathrm{e}, d \otimes 1\right) \to 0,$$

where $\pi'_{\overline{p}}(\alpha \otimes \mathrm{e}^n) = (\alpha, 0) \otimes \mathrm{e}^n$ and $\oint'_{\overline{p}}(\alpha, \beta) \otimes \mathrm{e}^n = \beta \otimes \mathrm{e}^n$. Each term is a differential complex and a Λe-module with the natural structure. Moreover, the maps $\pi'_{\overline{p}}$ and $\oint'_{\overline{p}}$ preserve these structures. The *equivariant Gysin sequence* is the induced long exact sequence

$$\cdots \to \left[\boldsymbol{H}^*_{\overline{p}}(B) \otimes \Lambda \mathrm{e}\right]^i \xrightarrow{\pi'_{\overline{p}}} \boldsymbol{H}^i_{\overline{p},\mathbb{S}^1}(X) \xrightarrow{\oint'_{\overline{p}}} \left[H^*\left(\mathcal{G}^{\cdot}_{\overline{p}}(B)\right) \otimes \Lambda \mathrm{e}\right]^{i-1} \xrightarrow{\delta_{\overline{p}}} \left[\boldsymbol{H}^*_{\overline{p}}(B) \otimes \Lambda \mathrm{e}\right]^{i+1} \to \cdots$$

Here, $\delta_{\overline{p}}([\beta] \otimes \mathrm{e}^n) = [d\alpha + (-1)^{|\beta|}\beta \wedge \epsilon] \otimes \mathrm{e}^n + (-1)^{|\beta|}\iota^B_{\overline{p}-\overline{x},\overline{p}}[\beta] \otimes \mathrm{e}^{n+1}$. For short, we shall write $(-1)^{|\beta|}\iota^B_{\overline{p}-\overline{x},\overline{p}} = I_{\overline{p}}$. The connecting morphism becomes $\delta_{\overline{p}} = \mathbf{e}_{\overline{p}} \otimes 1 + I_{\overline{p}} \otimes \mathrm{e}$. Notice that the equivariant Gysin sequence permits us to obtain the equivariant intersection cohomology of X in terms of basic[7] data.

3 Classification of modelled actions

In this section, we prove that B and the Euler class determine the equivariant intersection cohomology of X.

3.1 Fixing the orbit space[8]. Consider $\Phi_1 \colon \mathbb{S}^1 \times X_1 \to X_1$ and $\Phi_2 \colon \mathbb{S}^1 \times X_2 \to X_2$ two modelled actions and write B_1 and B_2 the corresponding orbit spaces.

An unfolded isomorphism $f \colon B_1 \to B_2$ is *optimal* when it preserves the nature of the strata. In this case, the two Euler perversities are equal: $\overline{e}_1(\pi_1(S)) = \overline{e}_2(f(\pi_1(S)))$ for each singular stratum S of X_1. We shall write $\overline{e}$ for this *Euler perversity*. Now we can compare the two Euler classes $e_1 \in \boldsymbol{H}^2_{\overline{e}}(B_1)$ and $e_2 \in \boldsymbol{H}^2_{\overline{e}}(B_2)$. We shall say that e_1 and e_2 are f*-related* if $f^*_{\overline{e}} e_2 = e_1$.

Proposition 3.2. *Let X_1, X_2 be two connected normal unfolded pseudomanifolds. Consider two modelled actions $\Phi_1 \colon \mathbb{S}^1 \times X_1 \to X_1$ and $\Phi_2 \colon \mathbb{S}^1 \times X_2 \to X_2$. Let us suppose that there exists an unfolded isomorphism $f \colon B_1 \to B_2$ between the associated orbit spaces. Then, the two following statements are equivalent:*

(a) The isomorphism f is optimal and the Euler classes e_1 and e_2 are f-related.

(b) There exists a Λe-perverse isomorphism $\boldsymbol{G} \colon \boldsymbol{H}_{\mathbb{S}^1}(X_2) \to \boldsymbol{H}_{\mathbb{S}^1}(X_1)$ verifying $\boldsymbol{G} \circ \boldsymbol{\pi}'_2 = \boldsymbol{\pi}'_1 \circ (\boldsymbol{f} \otimes 1)$.

Proof. We proceed in two steps.

$\boxed{(a) \Rightarrow (b)}$ Since $[f^*_{\overline{e}}\epsilon_2] = f^*_{\overline{e}} e_2 = e_1 = [\epsilon_1]$ then there exists $\gamma \in \Omega^1_{\overline{e}}(B_2)$ with $f^*_{\overline{e}}\epsilon_2 = \epsilon_1 - d(f^*_{\overline{e}}\gamma)$. Using this map γ, we construct for each perversity $\overline{p}$ the map $G_{\overline{p}} = F_{\overline{p}} \otimes 1 \colon \Omega^*_{\overline{p},\mathbb{S}^1}(X_2) \longrightarrow \Omega^*_{\overline{p},\mathbb{S}^1}(X_1)$ (cf.

[7] Of the orbit space B.

[8] See (cf. [10, sec. 3.1]) for details.

[10, Proposition 3.2]). Since $\boldsymbol{F} = \{F_{\overline{p}}\}$ is a perverse isomorphism, then $\boldsymbol{G} = \{G_{\overline{p}}\} \colon \boldsymbol{\Omega}_{\mathbb{S}^1}(X_2) \to \boldsymbol{\Omega}_{\mathbb{S}^1}(X_1)$ is a $\Lambda \mathsf{e}$-perverse isomorphism. The equality $\boldsymbol{G} \circ \boldsymbol{\pi}'_{\mathbf{2}} = \boldsymbol{\pi}'_{\mathbf{1}} \circ (\boldsymbol{f} \times 1)$ comes from:

$$G_{\overline{p}}\left(\pi'_{2,\overline{p}}(\alpha \otimes \mathsf{e}^n)\right) = F_{\overline{p}}(\alpha, 0) \otimes \mathsf{e}^n = (f_{\overline{p}}(\alpha), 0) \otimes \mathsf{e}^n = \pi'_{1,\overline{p}}\left(f_{\overline{p}}(\alpha) \otimes \mathsf{e}^n\right),$$

where $\alpha \in \Omega^*_{\overline{p}}(B_2)$.

$\boxed{(b) \Rightarrow (a)}$ Consider now the equivariant Gysin sequences associated to the actions Φ_1 and Φ_2. The two Gysin terms are written ${}_1\mathcal{G}$ and ${}_2\mathcal{G}$ respectively. Since $G_{\overline{e}_2} \circ \pi'_{2,\overline{e}_2} = \pi'_{1,\overline{e}_2} \circ (f_{\overline{e}_2} \times 1)$ we can construct the commutative diagram

$$\begin{array}{ccccccccc}
\boldsymbol{H}^1_{\overline{e}_2}(B_2) & \xrightarrow{\pi'_{2,\overline{e}_2}} & \boldsymbol{H}^1_{\overline{e}_2,\mathbb{S}^1}(X_2) & \xrightarrow{\oint'_{2,\overline{e}_2}} & H^0\left({}_1\mathcal{G}^*_{\overline{e}_2}(B_2)\right) & \xrightarrow{\delta'_{2,\overline{e}_2}} & \left[\boldsymbol{H}^*_{\overline{e}_2}(B_2) \otimes \Lambda \mathsf{e}\right]^2 & \xrightarrow{\pi'_{2,\overline{e}_2}} & \boldsymbol{H}^2_{\overline{e}_2,\mathbb{S}^1}(X_2) \\
\downarrow f_{\overline{e}_2} & & G_{\overline{e}_2}\downarrow & & \ell \downarrow & & \downarrow f_{\overline{e}_2} \otimes 1 & & G_{\overline{e}_2}\downarrow \\
\boldsymbol{H}^1_{\overline{e}_2}(B_1) & \xrightarrow{\pi'_{1,\overline{e}_2}} & \boldsymbol{H}^1_{\overline{e}_2,\mathbb{S}^1}(X_1) & \xrightarrow{\oint'_{1,\overline{e}_2}} & H^0\left({}_2\mathcal{G}^*_{\overline{e}_2}(B_1)\right) & \xrightarrow{\delta'_{1,\overline{e}_2}} & \left[\boldsymbol{H}^*_{\overline{e}_2}(B_1) \otimes \Lambda \mathsf{e}\right]^2 & \xrightarrow{\pi'_{1,\overline{e}_2}} & \boldsymbol{H}^2_{\overline{e}_2,\mathbb{S}^1}(X_1),
\end{array}$$

where $\ell \colon H^0\left({}_1\mathcal{G}^*_{\overline{e}_1}(B)\right) \to H^0\left({}_2\mathcal{G}^*_{\overline{e}_1}(B)\right)$ is an isomorphism. Following the proof of [10, Proposition 3.2] we conclude that the isomorphism f is optimal, the operator ℓ is the multiplication by a number $\lambda \in \mathbb{R}\backslash\{0\}$ and $f_{\overline{e}} e_2 = \lambda \cdot e_1$. Finally, the commutativity $(f_{\overline{e}} \otimes 1) \circ \delta'_{2,\overline{e}_2} = \delta'_{1,\overline{e}_2} \circ \ell$ gives that $\lambda = 1$ and therefore the Euler classes e_1 and e_2 are f-related. ♣

4 The basic spectral sequence

The Leray spectral sequence considered by Borel for the usual equivariant cohomology has been extended to the perverse framework in [1]. It converges to $\boldsymbol{H}^*_{\overline{p},\mathbb{S}^1}(X)$ and its second term is $\boldsymbol{H}^*_{\overline{p}}(X) \otimes \Lambda \mathsf{e}$.

We construct another spectral sequence converging to $\boldsymbol{H}^*_{\overline{p},\mathbb{S}^1}(X)$ whose third term is described in terms of B. It is the *basic spectral sequence*. First of all, we present an auxiliary complex.

4.1 The co-Gysin complex. The third term of the spectral sequence is described in terms of the *co-Gysin complex*[9] $\mathcal{K}^*_{\overline{p}}(B) = \dfrac{\Omega^*_{\overline{p}}(B)}{\mathcal{G}^*_{\overline{p}}(B)}$. It fits into the long exact sequence

$$\cdots \to H^i\left(\mathcal{G}^*_{\overline{p}}(B)\right) \xrightarrow{I_{\overline{p}}} \boldsymbol{H}^i_{\overline{p}}(B) \xrightarrow{P_{\overline{p}}} H^i\left(\mathcal{K}^*_{\overline{p}}(B)\right) \xrightarrow{\partial_{\overline{p}}} H^{i+1}\left(\mathcal{G}^*_{\overline{p}}(B)\right) \to \cdots.$$

Here, $I_{\overline{p}}[\alpha] = [\alpha]$, $\partial_{\overline{p}}[\overline{\alpha}] = [d\alpha]$ and $P_{\overline{p}}[\alpha] = [\overline{\alpha}]$. Now, we can describe the basic spectral sequence.

Proposition 4.2. *Consider a modelled action $\Phi \colon \mathbb{S}^1 \times X \to X$ and fix a perversity $\overline{p}$. There exists a first quadrant spectral sequence $\left\{\left(E_{\overline{p},r}, d_{\overline{p},r}\right)\right\}_{r \geq 0}$ converging to the equivariant intersection cohomology $\boldsymbol{H}^*_{\overline{p},\mathbb{S}^1}(X)$ such that*

(a) $E^{i,j}_{\overline{p},r} = 0$ if j is an odd number and $r \geq 1$;

[9] An element of $\mathcal{K}^*_{\overline{p}}(B)$ is written $\overline{\alpha}$ where $\alpha \in \Omega^*_{\overline{p}}(B)$.

(b) $E^{i,2j}_{\overline{p},2s} = E^{i,2j}_{\overline{p},2s+1}$ *if* $s \geq 1$*;*

(c) the second and third terms are $E^{i,2j}_{\overline{p},3} = E^{i,2j}_{\overline{p},2} = \begin{cases} \boldsymbol{H}^{i}_{\overline{p}}(B) & \text{if } j = 0 \\ H^{i}\left(\mathcal{K}^{*}_{\overline{p}}(B)\right) \otimes \mathbb{R} \cdot \mathsf{e}^{j} & \text{if } j > 0; \end{cases}$

(d) the third differential $d_{\overline{p},3} \colon E^{i,2j}_{\overline{p},3} \longrightarrow E^{i+3,2j-2}_{\overline{p},3}$ *is* $d_{\overline{p},3}(w \otimes \mathsf{e}^{j}) = \begin{cases} \left(\mathbf{e}_{\overline{p}} \circ \partial_{\overline{p}}\right)(w) & \text{if } j = 1 \\ \left(P_{\overline{p}} \circ \mathbf{e}_{\overline{p}} \circ \partial_{\overline{p}}\right)(w) \otimes \mathsf{e}^{j-1} & \text{if } j \geq 2. \end{cases}$

Proof. Consider the filtration $\cdots \subset F^{i}\Omega^{*}_{\overline{p},\mathbb{S}^1}(X) \subset F^{i-1}\Omega^{*}_{\overline{p},\mathbb{S}^1}(X) \subset \cdots \subset F^{0}\Omega^{*}_{\overline{p},\mathbb{S}^1}(X) = \Omega^{*}_{\overline{p},\mathbb{S}^1}(X)$ defined by $F^{i}\Omega^{*}_{\overline{p},\mathbb{S}^1}(X) = \{\omega \in \underline{\Omega}^{\geq i}_{\overline{p}}(X) \otimes \Lambda \mathsf{e} \ / \ d\omega \in \underline{\Omega}^{\geq i}_{\overline{p}}(X) \otimes \Lambda \mathsf{e}\}$. That is,

$$\begin{aligned} F^{i}\Omega^{i+2j}_{\overline{p},\mathbb{S}^1}(X) &= \left(\Omega^{i}_{\overline{p}}(B) \oplus \{0\}\right) \otimes \mathbb{R} \cdot \mathsf{e}^{j} \oplus \bigoplus_{k=1}^{j} \underline{\Omega}^{i+2k}_{\overline{p}}(X) \otimes \mathbb{R} \cdot \mathsf{e}^{j-k}, \text{ and} \\ F^{i}\Omega^{i+2j+1}_{\overline{p},\mathbb{S}^1}(X) &= \underline{\Omega}^{i+1}_{\overline{p}}(X) \otimes \mathbb{R} \cdot \mathsf{e}^{j} \oplus \bigoplus_{k=1}^{j} \underline{\Omega}^{i+1+2k}_{\overline{p}}(X) \otimes \mathbb{R} \cdot \mathsf{e}^{j-k} \end{aligned}$$

which verifies $\nabla\left(F^{i}\Omega^{*}_{\overline{p},\mathbb{S}^1}(X)\right) \subset F^{i}\Omega^{*+1}_{\overline{p},\mathbb{S}^1}(X)$. Following the standard procedure (see for example [8]) one constructs a spectral sequence $\left\{\left(E_{\overline{p},r}, d_{\overline{p},r}\right)\right\}$ converging to the equivariant cohomology $\boldsymbol{H}^{*}_{\overline{p},\mathbb{S}^1}(X)$. We have

$$E^{i,2j}_{\overline{p},0} = \left(\Omega^{i}_{\overline{p}}(B) \oplus \{0\}\right) \otimes \mathsf{e}^{j} \quad \text{and} \quad E^{i,2j+1}_{\overline{p},0} = \frac{\underline{\Omega}^{i+1}_{\overline{p}}(X)}{\Omega^{i+1}_{\overline{p}}(B) \oplus \{0\}} \otimes \mathsf{e}^{j}.$$

The differential $d_{\overline{p},0} \colon E^{i,2j}_{\overline{p},0} \to E^{i,2j+1}_{\overline{p},0}$ is zero, and the differential $d_{\overline{p},0} \colon E^{i,2j+1}_{\overline{p},0} \to E^{i,2j+2}_{\overline{p},0}$ is given by $d_{\overline{p},r}\left(\overline{(\alpha,\beta)} \otimes \mathsf{e}^{j}\right) = (\beta, 0) \otimes \mathsf{e}^{j+1}$. We conclude that $E^{i,j'}_{\overline{p},1} = \begin{cases} \Omega^{i}_{\overline{p}}(B) & \text{if } j' = 0 \\ \mathcal{K}^{i}_{\overline{p}}(B) \otimes \mathbb{R} \cdot \mathsf{e}^{j} & \text{if } j' = 2j > 0 \\ 0 & \text{if } j' \text{ is odd} \end{cases}$. This gives (a), $d_{\overline{p},2s} = 0$ if $s \geq 1$ and (b).

The first differential $d_{\overline{p},1} \colon E^{i,2j}_{\overline{p},1} \to E^{i+1,2j}_{\overline{p},1}$ is given by $d_{\overline{p},1}(\alpha) = d\alpha$ and $d_{\overline{p},1}\left(\overline{\alpha} \otimes \mathsf{e}^{j}\right) = d\overline{\alpha} \otimes \mathsf{e}^{j}$. We conclude that $E^{i,2j}_{\overline{p},3} = E^{i,2j}_{\overline{p},2} = \begin{cases} \boldsymbol{H}^{i}_{\overline{p}}(B) & \text{if } j = 0 \\ H^{i}\left(\mathcal{K}^{*}_{\overline{p}}(B)\right) \otimes \mathbb{R} \cdot \mathsf{e}^{j} & \text{if } j > 0. \end{cases}$. This gives (c).

Consider, for the computation of the third differential, $[\overline{\alpha}] \in H^{i}\left(\mathcal{K}^{*}_{\overline{p}}(B)\right)$. So, we have that $[d\alpha] \in H^{i+1}\left(\mathcal{G}^{*}_{\overline{p}}(B)\right)$ and $\mathbf{e}_{\overline{p}}([d\alpha]) \in \boldsymbol{H}^{i+3}_{\overline{p}}(B)$. This gives $d_{\overline{p},3}([\overline{\alpha}] \otimes \mathsf{e}) = \mathbf{e}_{\overline{p}}([d\alpha]) = \mathbf{e}_{\overline{p}}\partial_{\overline{p}}([\overline{\alpha}])$. For the general case $j \geq 2$ we have $d_{\overline{p},3}([\overline{\alpha}] \otimes \mathsf{e}^{j}) = P_{\overline{p}}\mathbf{e}_{\overline{p}}([d\alpha]) \otimes \mathsf{e}^{j-1} = P_{\overline{p}}\mathbf{e}_{\overline{p}}\partial_{\overline{p}}([\overline{\alpha}]) \otimes \mathsf{e}^{j-1}$. This gives (d). ♣

4.3 The basic spectral sequence in the classic framework. We consider here the usual cohomology, that is, the case $\overline{p} = \overline{0}$. For the sake of simplicity we also suppose that X is normal.

In this context, the basic spectral sequence is a spectral sequence converging to $H^{*}_{\mathbb{S}^1}(X)$ whose third term is described in terms of the cohomology of B and F, the union of fixed strata. In fact, as C. Allday pointed out to us, this spectral sequence degenerates into the Skjelbred exact sequence (cf. [13]).

First of all, we fix some facts. The cohomology of the complex $\mathcal{G}^{*}_{-\overline{x}}(B) = \Omega^{*}_{-\overline{e}}(B) = \Omega^{*}_{-\overline{x}}(B) = \mathcal{G}^{*}_{\overline{0}}(B)$ is $H^{*}(B, F)$ (cf. [12]). We shall write $\mathbf{e} \colon H^{*}(B, F) \to H^{*+2}(B, F)$ the map induced from $\mathbf{e}_{-\overline{x}}$. The long exact sequence associated to the pair (B, F) is $\cdots \to H^{i}(B, F) \xrightarrow{\iota} H^{i}(B) \xrightarrow{P} H^{i}(F) \xrightarrow{\partial} H^{i+1}(B, F) \to \cdots$.

Lemma 4.4. *Consider a modelled action $\Phi\colon \mathbb{S}^1\times X \to X$ where X is normal. There exists a first quadrant spectral sequence $\{(E_r, d_r)\}_{r\geq 0}$ converging to $H^*_{\mathbb{S}^1}(X)$ such that*

(a) $E_r^{i,j} = 0$ *if j is an odd number and $r \geq 1$;*

(b) $E_{2s}^{i,2j} = E_{2s+1}^{i,2j}$ *if $s \geq 1$;*

(c) $E_3^{i,2j} = E_2^{i,2j} = \begin{cases} H^i(B) & \text{if } j = 0 \\ H^i(F)\otimes\mathbb{R}\cdot \mathrm{e}^j & \text{if } j > 0; \end{cases}$

(d) *each $E_r^{*,0}$ is a quotient of $H^*(B)$ when $r \geq 3$;*

(e) *for each $s \geq 1$ and $j \neq s$, the differential $d_{2s+1}\colon E_{2s+1}^{i,2j} \to E_{2s+1}^{i+2s+1,2j-2s}$ is 0;*

(f) *for each $s \geq 1$, the differential $d_{2s+1}\colon E_{2s+1}^{i,2s} = H^i(F)\otimes\mathbb{R}\cdot\mathrm{e}^s \to E_{2s+1}^{i+2s+1,0}$ is induced by $(-1)^s\,\iota\circ\mathbf{e}^s\circ\partial$.*

Proof. Consider $\{(E_r, d_r)\}_{r\geq 0}$ the spectral sequence given by the above Proposition for $\overline{p} = \overline{0}$. It converges to $\boldsymbol{H}^*_{\overline{0},\mathbb{S}^1}(X)$ which is $H^*_{\mathbb{S}^1}(X)$ since X is normal. Let us verify the properties.

(a) and (b) Clear.

(c) Since X is normal, then B is normal and therefore $\boldsymbol{H}^*_{\overline{0}}(B) = H^*(B)$. The long exact sequence associated to the short exact sequence $0 \to \Omega^*_{-\overline{x}}(B) \to \Omega^*_{\overline{0}}(B) \to \mathcal{K}^*_{\overline{0}}(B) \to 0$ becomes

$$\cdots \to H^i(B,F) \xrightarrow{\iota} H^i(B) \xrightarrow{P_{\overline{0}}} H^i\left(\mathcal{K}^\cdot_{\overline{0}}(B)\right) \xrightarrow{\partial_{\overline{0}}} H^{i+1}(B,F) \to \cdots .$$

So, there exists an isomorphism $\xi\colon H^*\left(\mathcal{K}^\cdot_{\overline{0}}(B)\right) \to H^*(F)$ with $\partial\circ\xi = \partial_{\overline{0}}$ and $\xi\circ P_{\overline{0}} = P$. The result comes now directly from Proposition 4.2.

(d) For $r = 2s+1 \geq 3$ we have $E_{2s+1}^{i,0} = \dfrac{Z_{2s+1}^{i,0}}{B_{2s}^{i,0}} = \dfrac{\Omega^i(B)\cap d^{-1}(0)}{B_{2s}^{i,0}} = \dfrac{H^i(B)}{B_{2s}^{i,0}/d\Omega^{i-1}(B)}$.

To prove (e) and (f) , we proceed by induction on s. Taking $s = 1$, we have from Proposition 4.2 and the above identifications: $d_3(w\otimes \mathrm{e}^j) = \begin{cases} (\iota\circ\mathbf{e}\circ\partial)(w) & \text{if } j=1 \\ \left(\xi\circ P_{\overline{0}}\circ\iota\circ\mathbf{e}\circ\partial\right)(w)\otimes\mathrm{e}^{j-1} & \text{if } j\geq 2 \end{cases} = \begin{cases} (\iota\circ\mathbf{e}\circ\partial)(w) & \text{if } j=1 \\ 0 & \text{if } j\geq 2. \end{cases}$.

Let us now suppose that the result is true for $s' < s$. The case $j < s$ is straightforward by dimension reasons. Consider now $j \geq s$. The induction hypothesis and (b) give the isomorphism chain

$$\nabla_{j,s}\colon E_{2s+1}^{i,2j} = \frac{Z_{2s+1}^{i,2j}}{Z_{2s}^{i+1,2j-1} + B_{2s}^{i,2j}} \to E_{2s}^{i,2j} \to E_{2s-1}^{i,2j} \to \cdots \to E_2^{i,2j} \to H^i(F)\otimes\mathbb{R}\cdot\mathrm{e}^j,$$

with $\nabla_{j,s}\left(\overline{\omega_j = (\alpha_0,0)\otimes\mathrm{e}^j + \sum_{k=1}^{j}(\alpha_k,\beta_k)\otimes\mathrm{e}^{j-k}}\right) = \xi\,[\overline{\alpha_0}]\otimes\mathrm{e}^j$. On the other hand, we have that the definition of the differential $d_{2s+1}\colon E_{2s+1}^{i,2j} = \dfrac{Z_{2s+1}^{i,2j}}{Z_{2s}^{i+1,2j-1} + B_{2s}^{i,2j}} \longrightarrow \dfrac{Z_{2s+1}^{i+2s+1,2j-2s}}{Z_{2s}^{i+2s+2,2j-2s-1} + B_{2s}^{i+2s+1,2j-2s}}$ is

$$d_{2s+1}(\omega_j) = \overline{(d\alpha_s + (-1)^{|\beta_s|}\beta_s\wedge\epsilon + (-1)^{|\beta_{s+1}|}\beta_{s+1}, 0)\otimes\mathrm{e}^{j-s} + \sum_{k=s+1}^{j}(\alpha'_k,\beta'_k)\otimes\mathrm{e}^{j-k}},$$

and therefore $(\nabla_{j-s,s} \circ d_{2s+1})(\boldsymbol{\omega_j}) = \xi\left[\overline{d\alpha_s + (-1)^{|\beta_s|}\beta_s \wedge \epsilon + (-1)^{|\beta_{s+1}|}\beta_{s+1}}\right] \otimes \mathsf{e}^{j-s} = 0$ since $\beta_s \wedge \epsilon \in \Omega^*_{\overline{-x}}(B)$. This implies $d_{2s+1}(\boldsymbol{\omega_j}) = 0$ for $j > s$. It remains the case $j = s$. For $\boldsymbol{\omega_s} = \nabla^{-1}_{s,s}([\omega] \otimes \mathsf{e}^s) \in E^{i,2s}_{2s+1} = \dfrac{Z^{i,2s}_{2s+1}}{B^{i,2s}_{2s}}$ we have from (d)

$$\begin{aligned} d_{2s+1}(\boldsymbol{\omega_s}) &= \overline{[d\alpha_s + (-1)^{|\beta_s|}\beta_s \wedge \epsilon]} = \overline{\left[\sum_{k=0}^{s}(-1)^{s-k} d\alpha_k \wedge \epsilon^{s-k}\right]} = (-1)^s \overline{[d\alpha_0 \wedge \epsilon^s]} = \\ &= (-1)^s \overline{(\iota \circ \mathbf{e}^s \circ \partial_{\bar{0}})[\overline{\alpha_0}]} = (-1)^s \overline{(\iota \circ \mathbf{e}^s \circ \partial)\xi[\overline{\alpha_0}]} = (-1)^s \overline{(\iota \circ \mathbf{e}^s \circ \partial)[\omega]}, \end{aligned}$$

since $\displaystyle\sum_{k=1}^{s}(-1)^{s-k} d\alpha_k \wedge \epsilon^{s-k} = d\left(\sum_{k=1}^{s}\left(\alpha_k, (-1)^{|\alpha_k|}\sum_{j=1}^{k-1}(-1)^{k-1-j} d\alpha_j \wedge \epsilon^{k-1-j}\right) \otimes \mathsf{e}^{s-k}\right)$ belongs to $B^{i,0}_{2s}$. ♣

The particular geometry of this spectral sequence gives rise to a *Gysin sequence* (in the sense of [8]).

Proposition 4.5. *Consider a modelled action $\Phi\colon \mathbb{S}^1 \times X \to X$ where X is normal. We have the Skjelbred exact sequence*

$$\cdots \longrightarrow \left[H^*(F) \otimes \Lambda^{>0}\mathsf{e}\right]^i \xrightarrow{\boldsymbol{\beta}} H^{i+1}(B) \xrightarrow{\boldsymbol{\alpha}} \boldsymbol{H}^{i+1}_{\mathbb{S}^1}(X) \xrightarrow{\boldsymbol{\delta}} \left[H^*(F) \otimes \Lambda^{>0}\mathsf{e}\right]^{i+1} \xrightarrow{\boldsymbol{\beta}} H^{i+2}(B) \cdots,$$

where

- $\boldsymbol{\alpha}[\alpha] = [(\alpha, 0) \otimes 1]$;
- $\boldsymbol{\beta}([\omega] \otimes \mathsf{e}^s) = (-1)^s\,(\iota \circ \mathbf{e}^s \circ \partial)[\omega]$;
- $\boldsymbol{\delta}\left[\sum_{k=0}^{j}(\alpha_k, \beta_k) \otimes \mathsf{e}^k\right] = \sum_{k=1}^{j}\xi[\overline{\alpha_k}] \otimes \mathsf{e}^k$.

Proof. Consider the exact sequence $0 \longrightarrow \bigoplus_{s \geq 1} E^{i-2s,2s}_{\infty} \xrightarrow{\nabla} \bigoplus_{s \geq 1} H^{i-2s}(F) \otimes \mathsf{e}^s \xrightarrow{\boldsymbol{\beta}} H^{i+1}(B) \xrightarrow{\text{Proj}} E^{i+1,0}_{\infty} \longrightarrow 0$, where ∇ is induced by $\nabla_{s,s+1}\colon E^{i-2s,2s}_{\infty} = E^{i-2s,2s}_{2s+3} \to H^{i-2s}(F) \otimes \mathbb{R} \cdot \mathsf{e}^s$, and proceed as in [8, pag. 8]. ♣

5 Localization

The localization of the equivariant intersection cohomology is a cohomological theory introduced in [1, 6]. In fact, it is a residual cohomology since it depends on a neighborhood of the fixed point set F. The usual[10] LocalizationTheorem establishes that the localization $L^*_{\mathbb{S}^1}(X)$ of $H^*_{\mathbb{S}^1}(X)$ is in fact $H^*(F) \otimes \mathbb{R}(\mathsf{e})$. This doesn't hold for the generic case since the links of strata are no longer spheres.

[10] When X is a manifold, the family of strata $\mathcal{S}_X$ is reduced to the regular stratum.

5.1 Definition and properties. Denote by $\mathbb{R}(\mathsf{e})$ the field of fractions of $\Lambda\mathsf{e}$. The localization of the equivariant intersection cohomology is $\boldsymbol{L}^{\star}_{\overline{p},\mathbb{S}^1}(X) = \boldsymbol{H}^{*}_{\overline{p},\mathbb{S}^1}(X)\otimes_{\Lambda\mathsf{e}}\mathbb{R}(\mathsf{e})$. It is not a graded $\mathbb{R}(\mathsf{e})$-vector space over $\mathbb{Z}$ but over $\mathbb{Z}_2$ by: $\boldsymbol{L}^{\star}_{\overline{p},\mathbb{S}^1}(X) = \boldsymbol{L}^{even}_{\overline{p},\mathbb{S}^1}(X)\oplus\boldsymbol{L}^{odd}_{\overline{p},\mathbb{S}^1}(X) = \boldsymbol{H}^{even}_{\overline{p},\mathbb{S}^1}(X)\otimes_{\Lambda\mathsf{e}}\mathbb{R}(\mathsf{e})\oplus\boldsymbol{H}^{odd}_{\overline{p},\mathbb{S}^1}(X)\otimes_{\Lambda\mathsf{e}}\mathbb{R}(\mathsf{e})$. It verifies the following properties.

(a) The localization $\boldsymbol{L}^{\star}_{\overline{0},\mathbb{S}^1}(X)$ is the usual localization $L^{*}_{\mathbb{S}^1}(X)$ when X is normal.

(b) A perverse algebra is a *perverse superalgebra* when the coefficient ring is $\mathbb{R}(\mathsf{e})$ (instead of $\mathbb{R}$) and it is graded over $\mathbb{Z}_2$ (instead of over $\mathbb{Z}$). In the same manner, we introduce the notions of *perverse superalgebra morphism* and *perverse superalgebra isomorphism*. The quadruple $\boldsymbol{L}_{\mathbb{S}^1}(X) = \left(\bigoplus_{\overline{p}\in\mathcal{P}_X}\boldsymbol{L}_{\overline{p},\mathbb{S}^1}(X),\iota,\wedge,0\right)$ is a perverse superalgebra. On the other hand, the operator $\boldsymbol{\pi}''\colon \boldsymbol{H}(B)\otimes\mathbb{R}(\mathsf{e}) \to \boldsymbol{L}_{\mathbb{S}^1}(X)$, defined by $\pi''_{\overline{p}}(b\otimes P) = \pi'_{\overline{p}}(b\otimes 1)\otimes_{\Lambda\mathsf{e}}P$, is a perverse superalgebra morphism.

(c) The localization of the equivariant intersection cohomology is a residual cohomology: the inclusion induces a perverse superalgebra isomorphism $\boldsymbol{L}^{*}_{\overline{p},\mathbb{S}^1}(X)\cong\boldsymbol{L}^{*}_{\overline{p},\mathbb{S}^1}(U)$ where $U\subset X$ is any neighborhood of the fixed point set F. In fact, $\boldsymbol{L}^{*}_{\overline{p},\mathbb{S}^1}(X)$ can be seen as the global sections of a sheaf $\mathcal{H}_{\overline{p}}$ defined on F. This sheaf is *constructible*, that is, locally constant on each fixed stratum S. Its stalk is given by (2).

(d) Let us suppose that X is compact. Given two complementary perversities $\overline{p}$ and $\overline{q}$ the wedge product induces the Poincaré Duality isomorphism: $\boldsymbol{H}^{*}_{\overline{p},\mathbb{S}^1}(X)\cong\boldsymbol{H}^{\dim X-*}_{\overline{q},\mathbb{S}^1}(X)$. This gives the $\mathbb{R}(e)$-isomorphism: $\boldsymbol{L}_{\overline{p}}(X)\cong\boldsymbol{L}_{\overline{q}}(X)$ (cf. [1]). It preserves (resp. inverts) the superalgebra structure when $\dim X$ is even (resp. odd).

(e) The equivariant Gysin sequence can be written in the following way

$$\cdots\to\boldsymbol{H}^{*}_{\overline{p}}(B)\otimes\Lambda\mathsf{e}\xrightarrow{\pi'_{\overline{p}}}\boldsymbol{H}^{*}_{\overline{p},\mathbb{S}^1}(X)\xrightarrow{\oint'_{\overline{p}}}H^{*}\left(\mathcal{G}^{\cdot}_{\overline{p}}(B)\right)\otimes\Lambda\mathsf{e}\xrightarrow{\mathbf{e}_{\overline{p}}\otimes 1+I_{\overline{p}}\otimes\mathsf{e}}\boldsymbol{H}^{*}_{\overline{p}}(B)\otimes\Lambda\mathsf{e}\to\cdots,$$

which is a long exact sequence in the category of $\Lambda\mathsf{e}$-modules. Since localization is an exact functor, we get the *localized Gysin sequence*

$$\cdots\to\boldsymbol{H}^{*}_{\overline{p}}(B)\otimes\mathbb{R}(\mathsf{e})\xrightarrow{\pi''_{\overline{p}}}\boldsymbol{L}^{\star}_{\overline{p},\mathbb{S}^1}(X)\xrightarrow{\oint''_{\overline{p}}}H^{*}\left(\mathcal{G}^{\cdot}_{\overline{p}}(B)\right)\otimes\mathbb{R}(\mathsf{e})\xrightarrow{\mathbf{e}_{\overline{p}}\otimes 1+I_{\overline{p}}\otimes\mathsf{e}}\boldsymbol{H}^{*}_{\overline{p}}(B)\otimes\mathbb{R}(\mathsf{e})\to\cdots,$$

where $\oint''_{\overline{p}}([c]\otimes_{\Lambda\mathsf{e}}R) = \oint'_{\overline{p}}[c]\otimes_{\Lambda\mathsf{e}}R$. Thus, we get $\boldsymbol{L}^{\star}_{\overline{p},\mathbb{S}^1}(X)$ in terms of basic data.

The following result relates the Euler class with the localization of the equivariant intersection cohomology of X. It is obtained straightforwardly from Proposition 3.2.

Proposition 5.2. *Let X_1, X_2 be two connected normal unfolded pseudomanifolds. Consider two modelled actions $\Phi_1\colon\mathbb{S}^1\times X_1\to X_1$ and $\Phi_2\colon\mathbb{S}^1\times X_2\to X_2$. Let us suppose that there exists an unfolded isomorphism $f\colon B_1\to B_2$ between the associated orbit spaces. Then, the first following statement implies the second one:*

(a) The isomorphism f is optimal and the Euler classes e_1 and e_2 are f-related.

(b) There exists a perverse superalgebra isomorphism $\boldsymbol{K}\colon\boldsymbol{L}_{\mathbb{S}^1}(X_2)\to\boldsymbol{L}_{\mathbb{S}^1}(X_1)$ verifying $\boldsymbol{K}\circ\boldsymbol{\pi}''_2 = \boldsymbol{\pi}''_1\circ(\boldsymbol{f}\otimes 1)$.

The reciprocal to this Theorem does not hold: just consider the Hopf action on $\mathbb{S}^3$ and the action by multiplication on the second factor of $\mathbb{S}^2\times\mathbb{S}^1$. The Euler classes are different, but as the actions are free, both localizations vanish.

6 Examples

We illustrate the results of this work with some particular modelled actions $\Phi\colon \mathbb{S}^1 \times X \to X$. We present: (a) the Gysin and co-Gysin terms, (b) the equivariant intersection cohomology and (c) the localization of the equivariant intersection cohomology.

6.1 The pseudomanifold X is a manifold. Consider the case where $\overline{0} \le \overline{p} \le \overline{t}$[11].

(a) $H^*\left(\mathcal{G}^{\cdot}_{\overline{p}}(B)\right) = \boldsymbol{H}^*_{\overline{p}-\overline{e}}(B)$ (cf. [9, sec. 6.4]) and $H^*\left(\mathcal{K}^{\cdot}_{\overline{p}}(B)\right) = H^*_{\frac{\overline{p}}{\overline{p}-\overline{e}}}(B)$ (cf. [7]). In particular, $H^*\left(\mathcal{G}^{\cdot}_{\overline{0}}(B)\right) = H^*(B,F)$ and $H^*\left(\mathcal{K}^{\cdot}_{\overline{0}}(B)\right) = H^*(F)$.

(b) $\boldsymbol{H}^i_{\overline{p},\mathbb{S}^1}(X) = H^*_{\mathbb{S}^1}(X)$, $E^{i,0}_{\overline{p},2} = \boldsymbol{H}^i_{\overline{p}}(B)$ and $E^{i,2j}_{\overline{p},2} = \prod_{S\in\mathcal{S}_X} H^{i-2\left[\frac{\overline{p}(S)}{2}\right]}(S)\otimes \mathsf{e}^j$, for $j>0$.

(c) $\boldsymbol{L}^\star_{\overline{p},\mathbb{S}^1}(X) = H(F)\otimes\mathbb{R}(e) = L^*_{\mathbb{S}^1}(X)$.

6.2 Free action. These actions are characterized by the condition $\overline{e} = \overline{x} = \overline{0}$.

(a) $H^*\left(\mathcal{G}^{\cdot}_{\overline{p}}(B)\right) = \boldsymbol{H}^*_{\overline{p}}(B)$ and $H^*\left(\mathcal{K}^{\cdot}_{\overline{p}}(B)\right) = 0$.

(b) The basic spectral sequence degenerates at the second term and we have $\boldsymbol{H}^*_{\overline{p},\mathbb{S}^1}(X) = \boldsymbol{H}^*_{\overline{p}}(B)$. The $\Lambda\mathsf{e}$-module structure is given by $\mathsf{e}\cdot b = e\wedge b$. The perverse super structure comes from that of B.

(c) $\boldsymbol{L}^\star_{\overline{p},\mathbb{S}^1}(X) = 0$. We observe that neither the Euler class e nor the Euler perversity $\overline{e}$ are determined by the localization of the intersection cohomology.

6.3 Action without perverse strata. These actions are characterized by the condition $\overline{e} = \overline{x}$.

(a) We have $\mathcal{G}^*_{\overline{p}}(B) = \Omega^*_{\overline{p}-\overline{x}}(B)$ and $\mathcal{K}^*_{\overline{p}}(B) = \Omega^*_{\frac{\overline{p}}{\overline{p}-\overline{x}}}(B)$.

(b) $E^{i,2j}_{\overline{p},2} = \begin{cases} H^i_{\overline{p}}(B) & \text{if } j=0 \\ \boldsymbol{H}^i_{\frac{\overline{p}}{\overline{p}-\overline{x}}}(B)\otimes\mathsf{e}^j & \text{if } j>0 \end{cases}.$

(c) The perverse super structure comes from that of B.

6.4 The Euler class e is zero. In particular, all the fixed strata are non-perverse. We have

(a) $H^*\left(\mathcal{G}^{\cdot}_{\overline{p}}(B)\right) = \boldsymbol{H}^*_{\overline{p}-\overline{x}}(B)$ and $H^*\left(\mathcal{K}^{\cdot}_{\overline{p}}(B)\right) = H^*_{\frac{\overline{p}}{\overline{p}-\overline{x}}}(B)$.

(b) $\boldsymbol{H}^*_{\overline{p}}(X) = \boldsymbol{H}^*_{\overline{p}}(B)\oplus\boldsymbol{H}^{*-1}_{\overline{p}-\overline{x}}(B)$. The basic spectral sequence degenerates at the second term and we have $\boldsymbol{H}^*_{\overline{p},\mathbb{S}^1}(X) = \boldsymbol{H}^*_{\overline{p}}(B)\oplus\left\{\boldsymbol{H}^*_{\frac{\overline{p}}{\overline{p}-\overline{x}}}(B)\otimes\Lambda^{>0}\mathsf{e}\right\}$. The $\Lambda\mathsf{e}$-module struture is given by $\mathsf{e}\cdot(b_0,\overline{b_1}\otimes\mathsf{e}^n) = (0,\overline{b_0}\otimes\mathsf{e}+\overline{b_1}\otimes\mathsf{e}^{n+1})$.

(c) $\boldsymbol{L}^\star_{\overline{p},\mathbb{S}^1}(X) = \boldsymbol{H}^*_{\frac{\overline{p}}{\overline{p}-\overline{x}}}(B)\otimes\mathbb{R}(\mathsf{e})$. The perverse super structure comes from that of B.

[11] In this range the intersection cohomology of X coincides with its cohomology (see for example [12]).

6.5 Local calculation. Consider a chart (U,φ) of a fixed point x lying on a stratum S. The open subset U is $\mathbb{S}^1$-invariant and describes the local geometry near x. It can be equivariantly retracted by isomorphisms to cL_S, endowed with the action Φ_{L_S}. So, it is enough to consider the case $U = cL_S$. We have

$$\text{(a) } H^i\left(\mathcal{G}^{\cdot}_{\overline{p}}\left(U/\mathbb{S}^1\right)\right) = \begin{cases} H^i\left(\mathcal{G}^{\cdot}_{\overline{p}}\left(L_S/\mathbb{S}^1\right)\right) & \text{if } i \leq m-2 \\ \operatorname{Ker}\left\{\mathbf{e}_{\overline{p}} \colon H^i\left(\mathcal{G}^{\cdot}_{\overline{p}}\left(L_S/\mathbb{S}^1\right)\right) \longrightarrow \boldsymbol{H}^{i+2}_{\overline{p}}\left(L_S/\mathbb{S}^1\right)\right\} & \text{if } i = m-1 \\ 0 & \text{if } i \geq m. \end{cases}$$

(cf. [9, sec. 7.2]).

(b) The computation of $\boldsymbol{H}^*_{\overline{p},\mathbb{S}^1}(U)$ is achieved through the following step-by-step procedure. Let $\overline{q}$ the perversity defined by: $\overline{q} = \overline{p}$ on $U\backslash S$ and $\overline{q}(U\cap S) = \overline{p}(U\cap S)-1 = m-1$. We have $\boldsymbol{H}^i_{\overline{p},\mathbb{S}^1}(U) = \boldsymbol{H}^i_{\overline{q},\mathbb{S}^1}(U)$, for $i \neq m, m+1$, and the exact sequence

$$0 \to \boldsymbol{H}^m_{\overline{q},\mathbb{S}^1}(U) \to \boldsymbol{H}^m_{\overline{p},\mathbb{S}^1}(U) \to \boldsymbol{H}^m_{\overline{p}}(L_S) \otimes \Lambda \mathsf{e} \to \boldsymbol{H}^{m+1}_{\overline{q},\mathbb{S}^1}(U) \to \boldsymbol{H}^{m+1}_{\overline{p},\mathbb{S}^1}(U) \to 0$$

For example, when the action Φ_{L_S} is free we get that

$$\boldsymbol{H}^*_{\overline{p},\mathbb{S}^1}(U) = \boldsymbol{H}^{\leq m-1}_{\overline{p}}\left(L_S/\mathbb{S}^1\right) \oplus \left\{\boldsymbol{H}^m_{\overline{p}}\left(L_S/\mathbb{S}^1\right) \otimes \Lambda\mathsf{e}\right\} \oplus \left\{\frac{\boldsymbol{H}^{m-1}_{\overline{p}}\left(L_S/\mathbb{S}^1\right)}{\operatorname{Ker}\left\{\mathbf{e}_{\overline{p}} \colon \boldsymbol{H}^{m-1}_{\overline{p}}\left(L_S/\mathbb{S}^1\right) \to \boldsymbol{H}^{m+1}_{\overline{p}}\left(L_S/\mathbb{S}^1\right)\right\}} \otimes \Lambda^{>0}\mathsf{e}\right\}.$$

The $\Lambda\mathsf{e}$-product is induced by $\mathsf{e}\cdot(b_1, \overline{b_2}\otimes\mathsf{e}^n, b_3\otimes\mathsf{e}^n) = \begin{cases} (b_1 \wedge e, \overline{b_2} \otimes \mathsf{e}^{n+1}, b_3 \otimes \mathsf{e}^{n+1}) & \text{if } |b_1| \leq m-3 \\ (0, \overline{b_1 \wedge \mathsf{e}} \otimes 1 + \overline{b_2} \otimes \mathsf{e}^{n+1}, b_3 \otimes \mathsf{e}^{n+1}) & \text{if } |b_1| = m-2 \\ (0, \overline{b_2} \otimes \mathsf{e}^{n+1}, b_1 \otimes \mathsf{e} + b_3 \otimes \mathsf{e}^{n+1}) & \text{if } |b_1| = m-1. \end{cases}$

(c) We have a long exact sequence

$$\cdots \to \boldsymbol{L}^\star_{\overline{q},\mathbb{S}^1}(U) \to \boldsymbol{L}^\star_{\overline{p},\mathbb{S}^1}(U) \to \boldsymbol{H}^m_{\overline{p}}(L_S) \otimes \mathbb{R}(e) \to \boldsymbol{L}^\star_{\overline{q},\mathbb{S}^1}(U) \to \boldsymbol{L}^\star_{\overline{p},\mathbb{S}^1}(U) \to \cdots.$$

When Φ_{L_S} is free then

$$\text{(2)} \qquad \boldsymbol{L}^\star_{\overline{p},\mathbb{S}^1}(U) = \left\{\frac{\boldsymbol{H}^{m-1}_{\overline{p}}\left(L_S/\mathbb{S}^1\right)}{\operatorname{Ker}\left\{\mathbf{e}_{\overline{p}} \colon \boldsymbol{H}^{m-1}_{\overline{p}}\left(L_S/\mathbb{S}^1\right) \to \boldsymbol{H}^{m+1}_{\overline{p}}\left(L_S/\mathbb{S}^1\right)\right\}} \oplus \boldsymbol{H}^m_{\overline{p}}\left(L_S/\mathbb{S}^1\right)\right\} \otimes \mathbb{R}(\mathsf{e}).$$

The perverse super structure comes from that of $L_S/\mathbb{S}^1$.

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